

# Static Portfolio Choice

Guillaume Rabault

*In this section of the course, I review the static portfolio choice problem. The investor chooses a portfolio structure which is then left alone. The investment criterion is the expected utility of wealth at a terminal date. I briefly review specifications for the utility function together with risk aversion concepts. I look at the case of constant absolute risk aversion and normal returns, with or without labour income. I then introduce mean variance preferences, linking them to expected utility. Mean variance with and without a risk free asset is studied. The link between mean variance preferences and the expected returns/beta relationship is explained (the key ingredient of the CAPM). I then touch on the implementation problem.*

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## Timing

- Two periods:
  - portfolio decisions in  $t=0$
  - outcome observed in  $t=1$
- Outcomes in date 1 are uncertain as of date 0; they are described by random variables which we will identify in the notation using tildas
  - $x$ : particular outcome;  $\tilde{x}$ : random variable

## Instruments

- Instrument  $i$  with price  $p_i$  in period 0 gives right to pay-off  $\tilde{x}_i$  in period 1
- A cash instrument is an instrument with known date 1 pay-off as of date 0
- For risky assets,  $\tilde{x}$  is uncertain as of date 0
- I'll assume there are  $N$  risky assets ( $i = 1, \dots, N$ ) and potentially cash (the riskless asset), which will then have index 0
- The set of assets will be denoted by  $\mathcal{I}$ , with either  $\mathcal{I} = (1, \dots, N)$  (no riskless asset) or  $\mathcal{I} = (0, \dots, N)$  (with a riskless asset)

## Returns

- The return of an instrument with price  $p$  and pay-off  $\tilde{x}$  is:

$$\tilde{R} = \frac{\tilde{x}}{p}$$

- The **rate** of return is  $\tilde{r} = \tilde{R} - 1$
- The rate of return of cash is usually denoted  $r^f$ ; it is known as of date 0

## Investment and returns

- From investment to pay-off
- From  $t = 0$  to  $t = 1$ :

$$\begin{aligned} - \phi &\longrightarrow \tilde{R}\phi \\ - \phi &\longrightarrow (1 + \tilde{r})\phi \end{aligned}$$

## Portfolios

- Wealth in period 0 is  $w_0$
- The portfolio is invested in period 0; quantities  $(\theta_i)_{i \in \mathcal{I}}$  are purchased
- They need to satisfy:

$$\sum_{i \in \mathcal{I}} \theta_i p_i = w_0$$

- One can choose as control variables:
  - quantities  $(\theta_i)_{i \in \mathcal{I}}$
  - dollar amounts invested on instruments  $(\phi_i)_{i \in \mathcal{I}}$  with  $\phi_i = \theta_i p_i$
  - wealth shares  $(\pi_i)_{i \in \mathcal{I}}$ , with  $\pi_i = \phi_i / w_0$

## Budget constraints

- Quantities:

$$\sum_{i \in \mathcal{I}} \theta_i p_i = w_0$$

- Dollar amounts:

$$\sum_{i \in \mathcal{I}} \phi_i = w_0$$

- Wealth shares:

$$\sum_{i \in \mathcal{I}} \pi_i = 1$$

## Borrowing

- Borrowing is best understood as a negative position in cash:
  - from  $t = 0$  to  $t = 1$
  - $\phi = -d \longrightarrow -d(1 + r^f)$

## Accounting for future wealth

- for a given initial wealth  $w_0$ , a portfolio allocation leads to a random final wealth  $\tilde{w}$  with:
  - quantities:  $\tilde{w} = \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i$
  - invested amounts:  $\tilde{w} = \sum_{i \in \mathcal{I}} \phi_i \tilde{R}_i$
  - wealth shares:  $\tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$
  - It is sometimes useful to introduce at date 1 an exogenous income (amount to be received) or liability (amount to be paid)  $\tilde{y}$
  - $\tilde{w} = \tilde{y} + \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i$
  - $\tilde{w} = \tilde{y} + \sum_{i \in \mathcal{I}} \phi_i \tilde{R}_i$
  - $\tilde{w} = \tilde{y} + w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$

## Some return arithmetic

- Without liability, we have:
  - portfolio return:

$$\tilde{R}_p = \frac{\tilde{w}}{w_0} = \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$$

- portfolio rate of return:

$$\tilde{r}_p = \frac{\tilde{w}}{w_0} = \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i$$

(since  $\sum_{i \in \mathcal{I}} \pi_i = 1$ )

## The space of excess returns

- In the presence of a riskless asset, it is convenient to introduce excess returns versus the riskless rate:

$$\begin{aligned} \tilde{r}_p &= \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i \\ &= r^f + \sum_{i=1}^N \pi_i (\tilde{r}_i - r^f). \end{aligned}$$

- The choice variables are initially  $(\pi_i)_{i \in \mathcal{I}}$ , under the constraint  $\sum_{i \in \mathcal{I}} \pi_i = 1$ .
- In the excess return space, the choice variables are  $(\pi_i)_{i=1}^N$  to which no budget constraint applies since it is enforced by  $\pi_0 = 1 - \sum_{i=1}^N \pi_i$ .

## The portfolio problem

- Future wealth is a random variable, with a specific distribution
- The portfolio problem:
  - choose quantities (amounts, wealth shares) so as to obtain the best wealth distribution possible
- How do we compare random outcomes?
  - expected utility (Von Neumann Morgenstern - VNM) of outcome:  $E[u(\tilde{w})]$
  - the utility function embodies attitudes towards risk of the decision maker

## Some remarks

- **The optimization problem cannot have a solution if there are arbitrage opportunities**
- Reminder: an arbitrage is a way to generate a strictly positive pay-off without committing any funds
- The existence of a solution to a portfolio optimization problem thus guarantees the existence of a strictly positive stochastic discount factor (see below). We will see this principle in action

## Arbitrage, the law of one price and SDFs

- A stochastic discount factor is a random variable  $\tilde{m}$  such that for any pay-off  $\tilde{x}$ , the market price can be recovered:

$$p = E[\tilde{m}\tilde{x}].$$

- The law of one price is equivalent to the existence of a stochastic discount factor. The absence of arbitrage is equivalent to the existence of an almost everywhere strictly positive discount factor. Broadly speaking, strict positivity ensures that a (possibly synthetic) asset with strictly positive payoff cannot have a strictly negative price (this would be an arbitrage).
- In the return space, the above relationship reads:

$$E[\tilde{m}\tilde{R}] = 1.$$

- The expectation of the discount factor is linked to the risk free rate:

$$E[\tilde{m}](1 + r^f) = 1.$$

- In the excess return space, this reads:

$$E[\tilde{m}(\tilde{r} - r^f)] = 0.$$

- We thus have, in the presence of a risk free asset<sup>1</sup>:

$$E[\tilde{r}] - r^f = -R^f \text{cov}(\tilde{m}, \tilde{R}),$$

which describes the structure of risk premia across assets as a result of the covariances with the SDF.

## Reminder on utility functions (1)

- VNM utility functions are determined up to a linear transformation
- Absolute risk aversion:  $\alpha(w) = -u''(w)/u'(w)$
- Relative risk aversion:  $\rho(w) = w\alpha(w)$
- Risk tolerance:  $\tau(w) = 1/\alpha(w)$
- Additive certainty equivalent: for a centered distribution  $\tilde{\varepsilon}_a$  and an initial level of wealth  $w$ , find  $\pi_a(w, \tilde{\varepsilon}_a)$  such that:

$$u(w - \pi_a) = E[u(w + \tilde{\varepsilon}_a)].$$

- Multiplicative certainty equivalent: for a centered distribution  $\tilde{\varepsilon}_m$  and an initial level of wealth  $w$ , find  $\pi_m(w, \tilde{\varepsilon}_m)$  such that:

$$u(w(1 - \pi_m)) = E[u(w(1 + \tilde{\varepsilon}_m))].$$

## Reminder on utility functions (2)

- For small (centered) additive risks of variance  $\sigma^2$ :  $\pi_a \approx \frac{1}{2}\sigma_a^2\alpha(w)$
- For small (centered) multiplicative risks of variance  $\sigma^2$ :  $\pi_m \approx \frac{1}{2}\sigma_m^2\rho(w)$

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<sup>1</sup>Write the discount factor condition as:

$$E[\tilde{m}\tilde{R}] = E[(\tilde{m} - E[\tilde{m}] + E[\tilde{m}])\tilde{R}] = 1,$$

and use the fact:

$$E[(\tilde{m} - E[\tilde{m}])\tilde{R}] = \text{Cov}(\tilde{m}, \tilde{R}).$$

## Some important utility functions

- CARA:  $u(w) = -\exp(-\alpha w)$ 
  - range:  $\mathbb{R}$
  - absolute risk aversion:  $\alpha(w) = \alpha$

- CRRA:

$$u_\rho(w) = \frac{c^{1-\rho}}{1-\rho}, \rho \geq 0, \rho \neq 1,$$

$$u_\rho(w) = \log(w), \rho = 1,$$

- range  $\mathbb{R}_+^*$
- relative risk aversion:  $\rho(w) = \rho$

## CRRA utility functions - fig 1

### Utility functions and return distributions

- Utility functions often have a restricted domain (frequently: positive consumption)
- Assumptions on return distributions have to be consistent
- For example, CRRA models require  $\tilde{R} \geq 0$  i.e.  $\tilde{r} \geq -1$ . This assumption is sometimes called ‘limited liability’: the owner of an asset cannot end up having to transfer cash to the issuer.
- This is a problem mainly for discrete time models (or continuous times models where prices can jump)

### Absolute or relative?

- The key consideration is the dependence of risk attitudes vis-à-vis the level of wealth
  - intuition suggests people accept greater dollar risk as their wealth rises

## An important benchmark: CARA & normally distributed returns

- Note that with normal returns, returns can be arbitrarily negative (no limited liability). Accordingly, the range of the utility function is  $\mathbb{R}$ .
- I assume that there is no labor income
- $\pi = (\pi_i)_{i \in \mathcal{I}}$

$$\begin{aligned} \max_{\pi} E[-\exp(-\alpha \tilde{w})] \\ \text{s.t. :} \end{aligned}$$

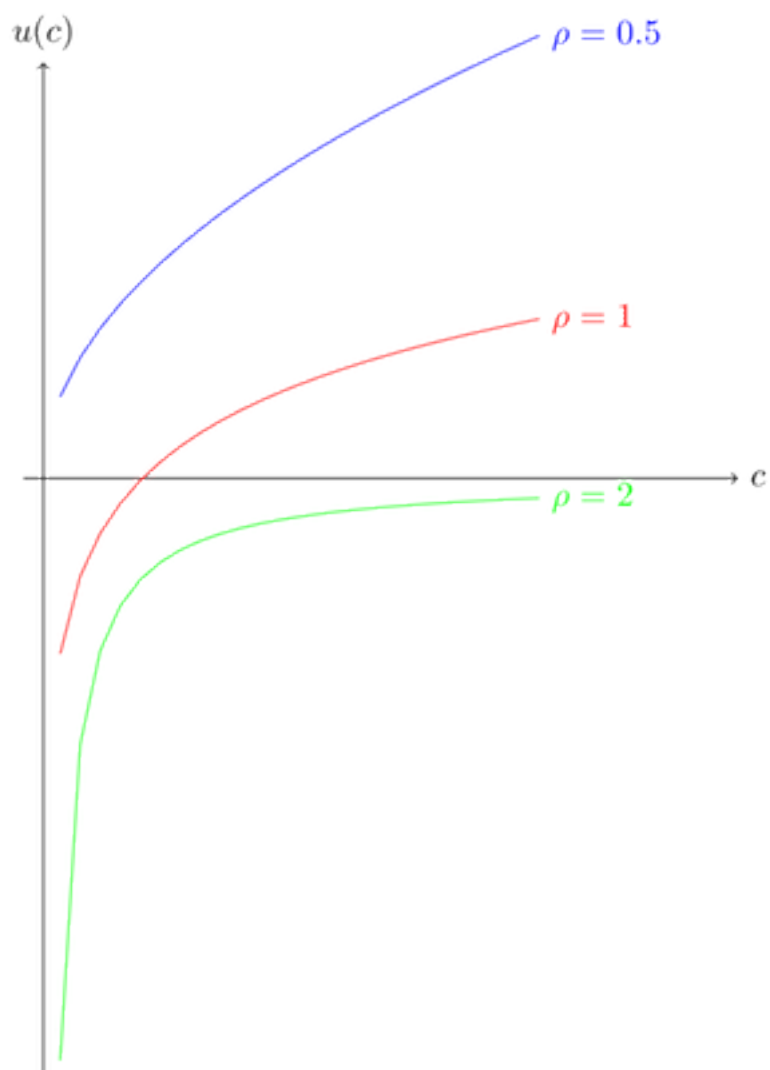


Figure 1: Figure 1: CRRA utility functions

$$\tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i$$

$$\sum_{i \in \mathcal{I}} \pi_i = 1.$$

### CARA normal case (1)

- The random variable  $\tilde{w}$  is normally distributed. In this case, we know that:

$$E[-\exp(-\alpha\tilde{w})] = -\exp(-\alpha E[\tilde{w}] + (\alpha^2/2)V[\tilde{w}])$$

$$= u(E[\tilde{w}] - (\alpha/2)V[\tilde{w}]).$$

- Given that the function  $u(\cdot)$  is increasing, the program consists in maximizing the certainty equivalent  $E[\tilde{w}] - (\alpha/2)V[\tilde{w}]$ , which reads, mean wealth minus the variance of wealth weighted by one half absolute risk aversion.

### CARA normal case (2)

- Preferences over the distribution of final wealth are thus entirely determined by the mean and the variance of the wealth distribution. This is an example of mean variance preferences.

- We have:

$$\tilde{w} = w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{R}_i = w_0 + w_0 \sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i$$

- The maximized criterion is thus (dividing by  $w_0 > 0$ ):

$$E[\sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i] - (\alpha w_0/2)V[\sum_{i \in \mathcal{I}} \pi_i \tilde{r}_i].$$

### CARA normal case (3)

- This is a standard mean-variance criterion, up to the fact that the risk aversion parameter depends on the level of wealth.
  - if this was not the case, optimal portfolio composition would be independent of the wealth level; this would imply that the investor take more dollar risk at higher wealth levels; in the CARA case, the appetite for dollar risk is independent of the level of wealth; thus the correction.

### When do we get mean variance preferences?

- How general is mean variance ?
  - preferences induced by utility functions will not, in general, correspond to mean-variance; additional assumptions are needed.



- when the distribution of portfolio returns is characterized by mean and variance, all utility functions naturally lead to mean variance preferences (see elliptic distributions).
- in the presence of stochastic labour income, mean variance needs to be amended

#### CARA normal case (4)

- In the presence of normally distributed stochastic labor income, the optimal programme is:

$$\begin{aligned} \max_{\boldsymbol{\pi}} \quad & E[-\exp(-\alpha\tilde{w})] \\ \text{s.t.} \quad & \\ & \tilde{w} = \tilde{y} + \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i \\ & \sum_{i \in \mathcal{I}} \theta_i p_i = w_0. \end{aligned}$$

- It is this time more convenient to take as control variables the quantities:  $(\theta_i)_{i \in \mathcal{I}}$ .

#### CARA normal case (5)

- As before, we need to maximize the certainty equivalent:  $E[\tilde{w}] - (\alpha/2)V[\tilde{w}]$ . This is equivalent to maximizing:

$$E \left[ \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i \right] - (\alpha/2)V \left[ \tilde{y} + \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i \right].$$

- We can decompose the variance term as:

$$V[\tilde{y}] + V \left[ \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i \right] + 2\text{Cov} \left( \sum_{i \in \mathcal{I}} \theta_i \tilde{x}_i, \tilde{y} \right).$$

#### CARA normal case (6)

- I give the result assuming there is a riskless asset.
- We assume the price of cash is  $p_0 = 1$ , and the payoff  $\tilde{x}_0 = 1 + r^f$ .
- Using the budget constraint  $\theta_0 = w_0 - \sum_{i=1}^N \theta_i p_i$ , we can rewrite the criterion as:

$$E \left[ w_0(1 + r^f) + \sum_{i=1}^N \theta_i (\tilde{x}_i - p_i(1 + r^f)) \right] - (\alpha/2)V \left[ \tilde{y} + \sum_{i=1}^N \theta_i \tilde{x}_i \right].$$

- Notation:
  - $\theta$  is the  $N \times 1$  vector of quantities invested on each risky asset
  - $V[\tilde{x}]$  is the  $N \times N$  matrix where each  $(i, j)$  is the covariance of the pay-offs of asset  $i$  and  $j$ . It is assumed to have full rank, so that no financial asset is riskless or redundant.
  - $\text{Cov}(\tilde{x}, \tilde{y})$  is the  $N \times 1$  vector where each entry measures the covariance of a financial instrument with labour income
  - $E[\tilde{x}]$  is the  $N \times 1$  vector of the expected excess pay-offs  $(\tilde{x}_i - p_i(1 + r^f))$  of the risky instruments.

### CARA normal case (7)

- The first order condition leads to, in matrix notation:

$$\theta = V[\tilde{x}]^{-1} \left( -\text{Cov}(\tilde{x}, \tilde{y}) + \frac{1}{\alpha} E[\tilde{x}] \right).$$

- Remember that  $1/\alpha$  is risk tolerance.
- The structure of the solution is as follows: the optimal portfolio consists of a hedging portfolio (which tries to replicate income variability using financial assets) and a speculative portfolio which has the same structure as in the case without labour income. The latter portfolio receives a weight equal to risk tolerance.

### Optimization and SDF

- I assume there is a solution  $\pi_*$  to the following problem:

$$\max_{\pi} E[u(\pi' \tilde{\mathbf{R}})]$$

s.t.

$$\pi' \mathbf{e} = 1,$$

where  $\mathbf{e}$  is a vector where all components are equal to 1, and  $\pi$  is the vector of asset proportions.

- The Lagrangian reads:

$$\mathcal{L} = E[u(\pi' \tilde{\mathbf{R}})] - \gamma \pi' \mathbf{e},$$

and the first order condition reads:

$$E[u'(\pi' \tilde{\mathbf{R}}) \tilde{\mathbf{R}}] = \gamma \mathbf{e}.$$

- Let:

$$\tilde{m} = \frac{u'(\pi'_* \tilde{\mathbf{R}})}{\gamma}.$$

We then have:

$$E[\tilde{m}\tilde{\mathbf{R}}] = \mathbf{e},$$

i.e. for any asset  $i$ :

$$E[\tilde{m}\tilde{R}_i] = 1.$$

In other words, we have built an SDF from the solution of the optimization problem.

### Mean variance efficiency

- A portfolio  $p$  with mean and variance  $(\mu_p, \sigma_p)$  is dominated by a portfolio  $q$  with mean and variance  $(\mu_q, \sigma_q)$  if  $\mu_q \geq \mu_p$  and  $\sigma_q \leq \sigma_p$  with at least one inequality being strict.
- A portfolio is efficient in the mean variance sense if it is not dominated by any other portfolio.
- Domination is a preorder. An efficient portfolio is a maximal element for the preorder. In particular, it is not a total order (all portfolio pairs cannot necessarily be ordered).

### Mean variance without a riskfree asset (1)

- The program: it consists in minimizing portfolio variance for a given level of expected returns

$$\begin{aligned} \min_{\boldsymbol{\pi}} V \left[ \sum_{i=1}^N \pi_i \tilde{r}_i \right] &= \boldsymbol{\pi}' \Sigma \boldsymbol{\pi} \\ \text{s.t.} \\ \sum_{i=1}^N \pi_i &= \boldsymbol{\pi}' \mathbf{e} = 1 \\ E \left[ \sum_{i=1}^N \pi_i \tilde{r}_i \right] &= \boldsymbol{\pi}' \boldsymbol{\mu} = \mu_p. \end{aligned}$$

### Mean variance without a riskfree asset (2)

- Bold notations denote vectors
  - $\Sigma$  is the covariance matrix of returns, which we assume invertible
  - $\mathbf{e}$  is a vector of ones
  - $\tilde{\mathbf{r}}$  is the vector of returns
  - $\boldsymbol{\mu}$  is the vector of expected returns
- We assume  $\boldsymbol{\mu} \neq \mathbf{e}$  to avoid degeneracy

### Mean variance without a riskfree asset (3)

- Lagrangian for the optimization problem (a factor 1/2 is convenient):

$$\frac{1}{2}\boldsymbol{\pi}'\Sigma\boldsymbol{\pi} - \delta(\boldsymbol{\pi}'\boldsymbol{\mu} - \mu_p) - \gamma(\boldsymbol{\pi}'\mathbf{e} - 1)$$

where I have introduced the Lagrange multipliers  $\delta$  and  $\gamma$ .

- The necessary and sufficient first order condition (positive definite quadratic problem) is:

$$\Sigma\boldsymbol{\pi} = \delta\boldsymbol{\mu} + \gamma\mathbf{e},$$

or, assuming the covariance matrix is invertible:

$$\boldsymbol{\pi} = \delta\Sigma^{-1}\boldsymbol{\mu} + \gamma\Sigma^{-1}\mathbf{e}.$$

### Mean variance without a riskfree asset (4)

- Injecting this into the constraints leads to a system for the Lagrange multipliers:

$$\delta\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu} + \gamma\boldsymbol{\mu}'\Sigma^{-1}\mathbf{e} = \mu_p,$$

$$\delta\mathbf{e}'\Sigma^{-1}\boldsymbol{\mu} + \gamma\mathbf{e}'\Sigma^{-1}\mathbf{e} = 1.$$

- Reminder:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- It is useful to introduce two specific portfolios:

$$\boldsymbol{\pi}_1 = \frac{1}{\mathbf{e}'\Sigma^{-1}\mathbf{e}}\Sigma^{-1}\mathbf{e},$$

$$\boldsymbol{\pi}_\mu = \frac{1}{\mathbf{e}'\Sigma^{-1}\boldsymbol{\mu}}\Sigma^{-1}\boldsymbol{\mu}.$$

### Mean variance without a riskfree asset (5)

- We can write :

$$\begin{aligned} \boldsymbol{\pi} &= (\delta\mathbf{e}'\Sigma^{-1}\boldsymbol{\mu})\boldsymbol{\pi}_\mu + (\gamma\mathbf{e}'\Sigma^{-1}\mathbf{e})\boldsymbol{\pi}_1 = \\ &\lambda\boldsymbol{\pi}_\mu + (1 - \lambda)\boldsymbol{\pi}_1. \end{aligned}$$

- Thus, any optimal portfolio is a combination of the two portfolios we singled out:

- $\boldsymbol{\pi}_1$  is the minimum variance portfolio
- $\boldsymbol{\pi}_\mu$  is another portfolio as soon as  $\boldsymbol{\mu} \neq \mathbf{e}$

### Mean variance without a riskfree asset (6)

- $A = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}$ ,  $B = \boldsymbol{\mu}'\Sigma^{-1}\mathbf{e}$ ,  $C = \mathbf{e}'\Sigma^{-1}\mathbf{e}$ .

$$\lambda = \frac{BC\mu_p - B^2}{AC - B^2},$$

$$\sigma_p^2 = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}.$$

- Check this.
- The efficient frontier (in the standard deviation mean space) is the subset of non dominated portfolios in the set:

$$\{(\sigma_p, \mu_p), \mu_p \geq \mu_1\}$$

where  $\mu_1 = \boldsymbol{\pi}'_1\boldsymbol{\mu}$ .

### Mean variance without a riskfree asset (7) - fig 2

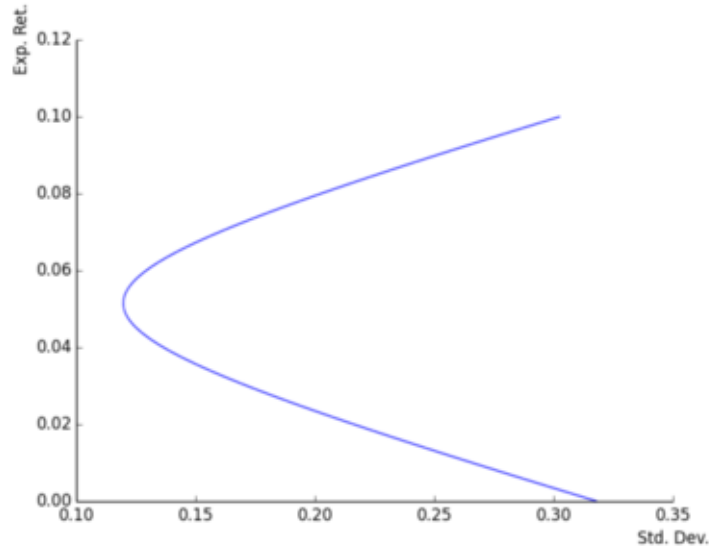


Figure 2: Figure 2: Efficient Frontier (without a risk free asset)

### Mean variance without a riskfree asset (8)

- I list the technical conditions below:
  - we assume that  $\boldsymbol{\mu}$  and  $\mathbf{e}$  are not colinear

- we assume  $\mathbf{e}'\Sigma^{-1}\boldsymbol{\mu} > 0$
- we have  $\mathbf{e}'\Sigma^{-1}\mathbf{e} > 0$  as  $\Sigma^{-1}$  defines a positive definite quadratic form
- we have  $(\boldsymbol{\mu}'\Sigma^{-1}\mathbf{e})^2 < (\mathbf{e}'\Sigma^{-1}\mathbf{e})(\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu})$  from the Cauchy-Schwartz inequality and  $\mathbf{e}'\Sigma^{-1}\mathbf{e} > 0$ .

### Mean variance with a riskfree asset (1)

- It is convenient in this case to use the notation  $\boldsymbol{\pi}$  to denote the vector of positions on the risky assets (see the slide on the space of excess returns). The cash position is thus:

$$\pi_0 = 1 - \mathbf{e}'\boldsymbol{\pi}.$$

- The vector  $\boldsymbol{\pi}$  is unconstrained. The optimization problem can be written:

$$\min_{\boldsymbol{\pi}} \boldsymbol{\pi}'\Sigma\boldsymbol{\pi}$$

s.t.

$$\boldsymbol{\pi}'(\boldsymbol{\mu} - r^f\mathbf{e}) = \mu_p - r^f.$$

- For reasons that will be clear below, I assume  $\mathbf{e}'\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e}) > 0$ .

### Mean variance with a riskfree asset (2)

- First order condition for the Lagrangian:  $\boldsymbol{\pi} = \delta\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e})$
- From  $(\boldsymbol{\mu} - r^f\mathbf{e})'\boldsymbol{\pi} = \mu_p - r^f$ , we get the value of  $\delta$  and then the value of  $\boldsymbol{\pi}$ :

$$\boldsymbol{\pi} = \frac{\mu_p - r^f}{(\boldsymbol{\mu} - r^f\mathbf{e})'\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e})}\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e}).$$

- The standard deviation of the portfolio is:

$$\frac{|\mu_p - r^f|}{\sqrt{(\boldsymbol{\mu} - r^f\mathbf{e})'\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e})}}.$$

### Mean variance with a riskfree asset (3)

- The tangency portfolio is:

$$\boldsymbol{\pi}_* = \frac{1}{\mathbf{e}'\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e})}\Sigma^{-1}(\boldsymbol{\mu} - r^f\mathbf{e}).$$

- It is a portfolio fully invested in risky assets which is on the overall efficient frontier. It is thus also on the risky asset efficient frontier.

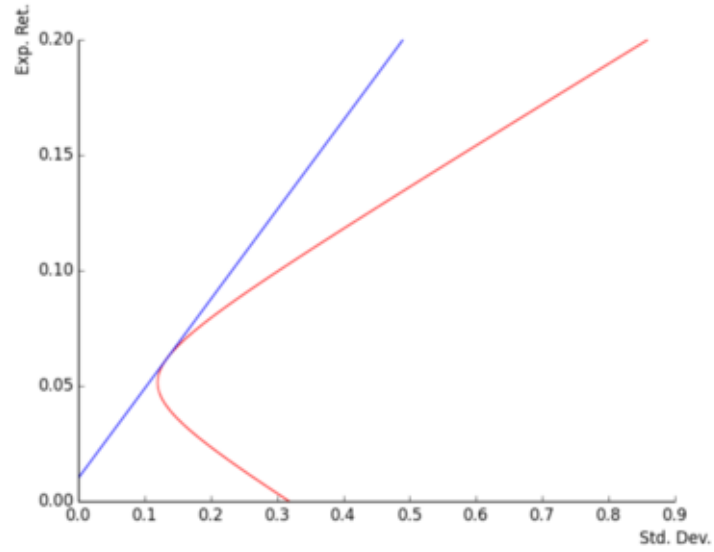


Figure 3: Figure 3: Efficient Frontier (with risk free asset)

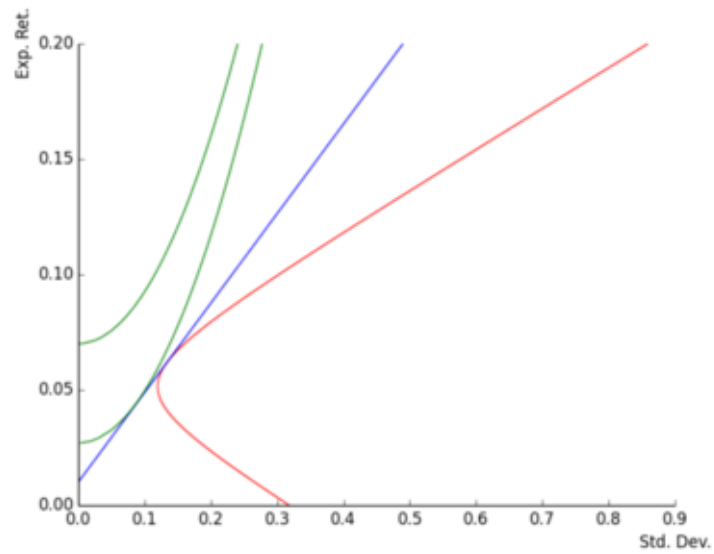


Figure 4: Figure 4: Efficient Frontier (with risk free asset): iso-utility curves

Mean variance with a riskfree asset (4) - fig 3

Mean variance with a riskfree asset (4) - fig4

Data for the graphs (1)

- Two risky assets:
  - $\mu_1 = 0.05, \sigma_1 = 0.12$
  - $\mu_2 = 0.07, \sigma_2 = 0.16$
  - $\rho = 0.7$
  - $(\mu_1 - r)/\sigma_1 = 0.33$
  - $(\mu_2 - r)/\sigma_2 = 0.375$
  - $\pi_1 = (0.93, 0.07)$
  - $\text{vol}(\pi_1) = 0.12$
  - $\pi_* = (0.4, 0.6)$
  - $\text{vol}(\pi_*) = 0.13$
  - $\text{sharpe}(\pi_*) = 0.39$

Data for the graphs (2)

- The graphs shown assume positive Sharpe ratios for the underlying assets. This is the ‘normal’ situation. It ensures that the efficient frontier (with a riskfree asset!) is upward sloping.

A different description of the efficient frontier (1)

- Maximize the expected return penalized for portfolio variance ( $\rho > 0$ ):

$$\max_{\pi} r^f + \pi'(\mu - r^f \mathbf{e}) - \frac{\rho}{2} \pi' \Sigma \pi.$$

- Exercise: recover the lagrange multiplier of the traditional approach
- The criteria are given by quadratic utility functions, indexed by  $\rho$

A different description of the efficient frontier (2)

- The first order condition reads:

$$(\mu - r^f \mathbf{e}) = \rho \Sigma \pi,$$

and this implies that the optimal portfolio is proportional to the tangency portfolio.

- How much of the tangency portfolio  $\pi_*$  does an investor with the above preferences and beliefs buy?



- From the first order condition of the utility maximization problem<sup>2</sup>, we get that the weight  $\hat{\pi} = 1 - \pi_0$  invested in the tangency portfolio is:

$$\hat{\pi} = \frac{1}{\rho} \frac{\mu_* - r^f}{\text{var}(\tilde{r}_*)}.$$

- We will remember that:

$$\rho\hat{\pi} = \frac{\mu_* - r^f}{\text{var}(\tilde{r}_*)},$$

which is therefore independent of the risk aversion level of the investor. This will play a role in the derivation of the CAPM.

### Interpretation of the first order condition (1)

- Consider that the optimal portfolio of a mean-variance investor ( $p$  with weights  $\boldsymbol{\pi}$ ) is tilted by adding a long-short portfolio  $\boldsymbol{\pi}_\delta$ . How does that affect quadratic utility?
- The utility level changes by (first order approximation):

$$\begin{aligned} & \mu_\delta - \rho \text{cov}(\tilde{r}_\delta, \tilde{r}_p) \\ &= \mu_\delta - \rho \frac{\text{cov}(\tilde{r}_\delta, \tilde{r}_p)}{\text{var}(\tilde{r}_p)} \text{var}(\tilde{r}_p), \\ &= \mu_\delta - \rho \beta(\tilde{r}_\delta, \tilde{r}_p) \text{var}(\tilde{r}_p), \\ &= \mu_\delta - \rho \hat{\pi} \beta(\tilde{r}_\delta, \tilde{r}_*) \text{var}(\tilde{r}_*). \end{aligned}$$

### Interpretation of the first order condition (2)

- Because the quantity  $\rho\hat{\pi}$  is independent of  $\rho$ , the trade off between return and beta is a well defined consequence of the mean and variance assumptions.
- Injecting the value of  $\rho\hat{\pi}$  into the first order condition delivers the quantity:

$$\mu_\delta - (\mu_* - r^f) \beta(\tilde{r}_\delta, \tilde{r}_*).$$

- Given the optimality of the tangency portfolio, the above quantity should be zero for all long short deviations to the tangency portfolio:

$$\mu_\delta = (\mu_* - r^f) \beta(\tilde{r}_\delta, \tilde{r}_*).$$

- For long short portfolios which borrow to buy a stock, the condition reads:

$$(\mu_i - r^f) = (\mu_* - r^f) \beta(\tilde{r}_i, \tilde{r}_*).$$

---

<sup>2</sup>The first order condition reads:

$$(\boldsymbol{\mu} - r^f \mathbf{e}) = \rho \Sigma \boldsymbol{\pi}.$$

Multiply both sides on the left by  $\boldsymbol{\pi}'$ . Then use  $\boldsymbol{\pi} = \hat{\pi} \boldsymbol{\pi}_*$ .

### Interpretation of the first order condition (3)

- The above relationship embodies the return beta trade off embedded in the mean variance assumptions.
- At this stage, no equilibrium assumption has been made. We are looking at the implications of a portfolio being mean-variance optimal.
- Note that the tangency portfolio can be replaced by any other efficient portfolio in the relationship.

### The excess return-beta relationship (1) - fig 5

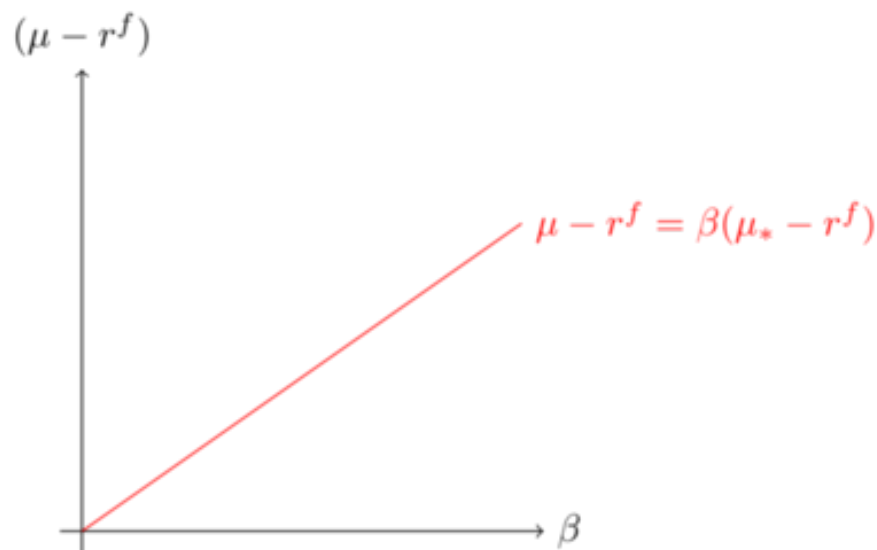


Figure 5: Figure 5: Return/beta relationship

### The excess return-beta relationship (2) - fig 6

#### The two fund theorem and the CAPM

- We now move to equilibrium considerations. We assume all investors share the same beliefs on expected returns and risk, and all choose mean variance efficient portfolios.
- As a result, they all hold a mixture of the risk free asset and a unique portfolio of risky asset, the tangency portfolio.
- This is an instance of the two fund theorem, which also holds in more general contexts

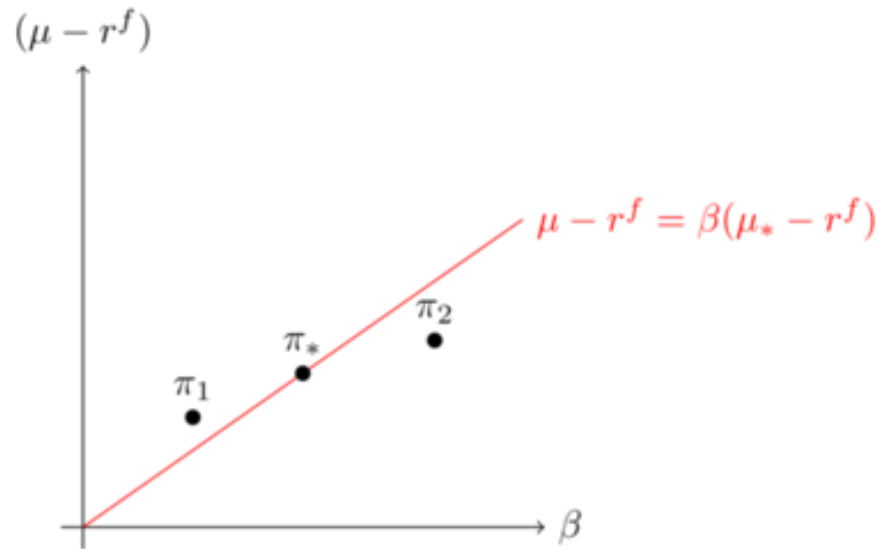


Figure 6: Imperfectly priced portfolios

- The risky asset portfolio should be equal to the market portfolio of risky asset, with return  $r_m$ . This gives:

$$(\mu_i - r^f) = (\mu_m - r^f)\beta(\tilde{r}_i, \tilde{r}_m).$$

## Illustration of the CAPM - fig 7

### The low beta anomaly - fig 8

### Equity pricing anomalies

- Take an investment universe (stocks) and an equity index
- Follow the steps:
  - build equity portfolios by sorting stocks according to a financial characteristic
  - compute the beta of the portfolios and graph realized returns against betas
  - is the pricing error significant?
- Examples of characteristics: size, book value, momentum, beta, vol
- This procedure asks whether the index is mean variance efficient in sample
- The pricing errors should be statistically significant

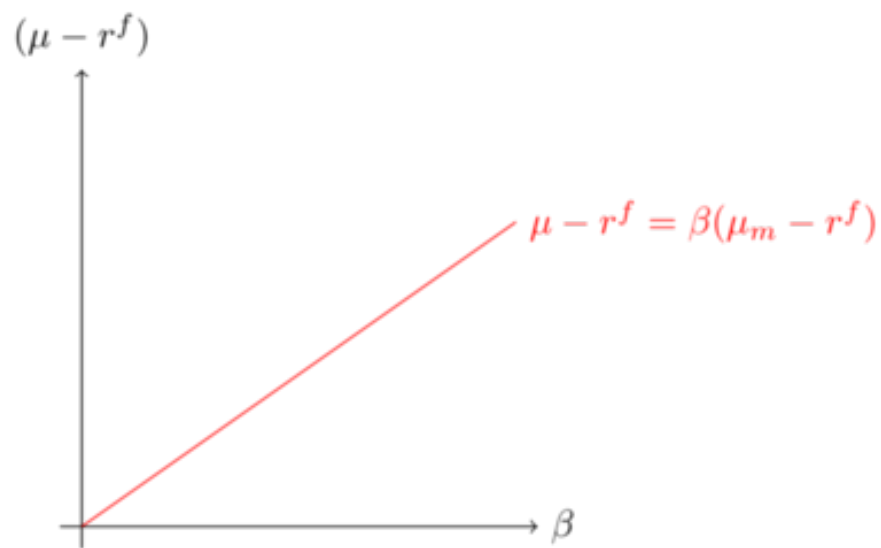


Figure 7: Figure 7: CAPM

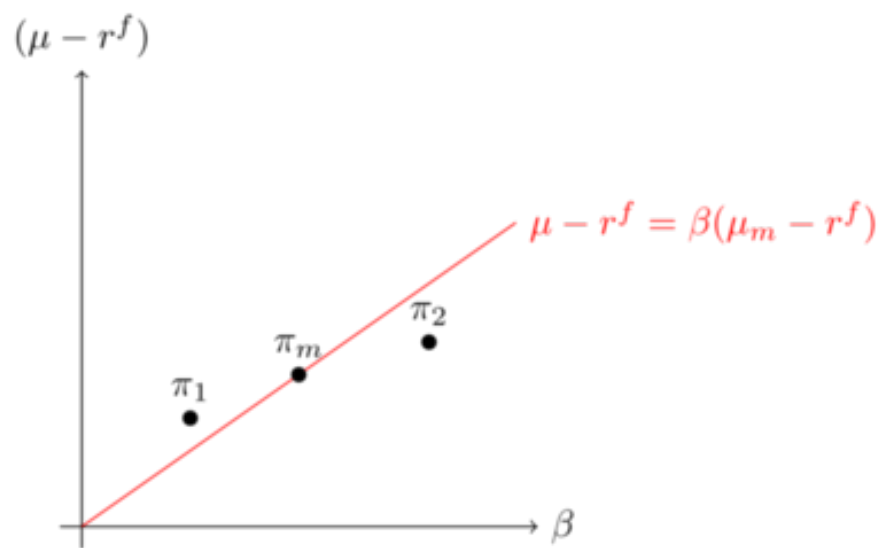


Figure 8: Figure 8: The low beta anomaly

## Mean variance in practice: the challenges

- First, there is a question of interpretation: what is the investment horizon?
  - in particular, this conditions the nature of the risky asset (bonds or cash).
- One also needs to be clear on whether real returns or nominal returns are considered
- Once this has been clarified, input data needs to be estimated:
  - getting hold of expected returns
  - getting hold of the covariance matrix
- The optimal portfolio is very sensitive to inputs
  - garbage in, garbage out

## Examples of implementations

- Give the same return to all risky assets
  - this delivers the minimum variance portfolio, which is not optimal unless the expected returns are truly equal across assets
- Link the return assumptions to the risk estimates
  - this leads to various solutions...
- For returns: estimate the payoffs of the asset and derive the implied return from the current asset price
- Example: ERC ?

## Links

- [Link to pdf](#)